

Gravitational solitons, hairy black holes and phase transitions in BHT massive gravity

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Abstract

Hairy black holes and gravitational solitons in three dimensions for the new massive gravity theory proposed by Bergshoeff, Hohm and Townsend (BHT) are considered at the special case when there is a unique maximally symmetric solution. Following the Brown-York approach with suitable counterterms, it is shown that the soliton possesses a fixed negative mass which coincides with the one of AdS spacetime regardless the value of the integration constant that describes it. Hence, the soliton can be naturally regarded as a degenerate ground state labeled by a single modulus parameter. The Euclidean action, endowed with suitable counterterms, is shown to be finite and independent of modulus and hair parameters for both classes of solutions, and in the case of hairy black holes the free energy in the semiclassical approximation is reproduced. Modular invariance allows to show that the gravitational hair turns out to be determined by the modulus parameter. According to Cardy's formula, it is shown that the semiclassical entropy agrees with the microscopic counting of states provided the modulus parameter of the ground state is spontaneously fixed, which suggests that the hairy black hole is in a broken phase of the theory. Indeed, it is found that there is a critical temperature $T_c = (2\pi l)^{-1}$ characterizing a first order phase transition between the static hairy black hole and the soliton which, due to the existence of gravitational hair, can take place in the semiclassical regime.

I. INTRODUCTION

Three-dimensional gravity has recently received a great deal of attention, specially in the case of extensions of General Relativity that admit propagating massive gravitons. Apart from the well-known theory of topologically massive gravity [1, 2], a different theory currently known as “new massive gravity” has been proposed by Bergshoeff, Hohm and Townsend (BHT) [3, 4]. The theory is described by the following action

$$I_{\text{BHT}} = \frac{1}{16\pi G} \int_M d^3x \sqrt{-g} \left[R - 2\lambda - \frac{1}{m^2} \left(R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right) \right], \quad (1)$$

which is manifestly invariant under parity and gives fourth order field equations for the metric. Remarkably, at the linearized level the field equations reduce to the ones of Fierz and Pauli for a massive graviton, generically describing two independent propagating degrees of freedom. This has also been confirmed at the full nonlinear level from different canonical approaches [5–7]. A wide variety of exact solutions can be found in Refs. [4, 13–25], and further aspects of the theory have been developed in [26–30]. Hereafter we will focus in the special case

$$m^2 = \lambda, \quad (2)$$

since the theory possesses additional interesting features. In this case the field equations admit a unique maximally symmetric solution (with $R^{\mu\nu}_{\alpha\beta} = \Lambda \delta^{\mu\nu}_{\alpha\beta}$) whose curvature radius is determined by $\Lambda = 2m^2$, and at the linearized level the graviton becomes “partially massless” [4, 8–12]. The special case (2) enjoys further remarkable properties, as it is the existence of interesting solutions including hairy black holes and gravitational solitons [13]. Hereafter we will focus on these latter solutions in the case of negative cosmological constant $\Lambda := -\frac{1}{\ell^2}$. We show that the soliton possesses a negative fixed mass which agrees with that of AdS, so that its integration constant turns out to be a modulus parameter. The regularized Euclidean action becomes independent of modulus and hair parameters for both classes of solutions, and it reduces to the free energy in the case of hairy black holes. An interesting link between both solutions can be seen through modular invariance, which allows to find a precise relationship between the gravitational hair of the black hole and the modulus parameter of the soliton. Noteworthy, it is found that this relation exactly maps the mass bound for the hairy black hole required by cosmic censorship with the condition that guarantees the regularity of the soliton. Furthermore, the semiclassical entropy agrees

with the microscopic counting of states according to Cardy formula provided the modulus parameter of the ground state is spontaneously fixed, which suggests that the hairy black hole is in a broken phase of the theory. Indeed, it is found that there is a critical temperature that characterizes a first order phase transition between the static hairy black hole and the soliton. Remarkably, the presence of gravitational hair induces an additional effective length scale which allows a suitable treatment of this phase transition in the semiclassical regime.

Our paper is organized as follows. In the next section the gravitational soliton is briefly reviewed, while its mass is obtained through the Brown-York approach with suitable counterterms in Sec. II A. The regularized Euclidean action is also analyzed in the case of the rotating hairy black hole in Sec. III, where it is shown to be independent of the hair parameter, and reduces to the free energy in the semiclassical approximation. Modular invariance is discussed in Sec. IV, which allows to show that the gravitational hair of the black hole turns out to be determined by the modulus parameter of the soliton. In Sec. IV A it is shown that the semiclassical entropy agrees with the microscopic counting of states provided the ground state is non degenerate. As discussed in Sec. V a first order phase transition between the static hairy black hole and the soliton is shown to occur at a critical temperature $T_c = (2\pi l)^{-1}$, while in Sec. V A we explain how the existence of gravitational hair may allow the transition to take place in the semiclassical regime. Finally, Sec. VI is devoted to the summary and discussion.

II. SOLITON AS A DEGENERATE GROUND STATE

It has been shown that the field equations that correspond to the action (1) at the special point (2), in the case of negative cosmological constant admit a solution whose metric is given by [13]

$$ds^2 = l^2 \left[-(\alpha + \cosh \rho)^2 dt^2 + d\rho^2 + \sinh^2 \rho d\varphi^2 \right] . \quad (3)$$

If the integration constant α fulfills

$$\alpha > -1 , \quad (4)$$

this spacetime is smooth and regular everywhere and describes a gravitational soliton. Note that in the case of $\alpha = 0$, the soliton (3) reduces to AdS spacetime in global coordinates. The soliton can then be regarded as a smooth deformation of AdS spacetime, sharing the same causal structure. Furthermore, the metric (3) is asymptotically AdS in a relaxed sense

as compared with the one of Brown and Henneaux [31], but nevertheless the asymptotic symmetries remains the same. Indeed, the soliton can be written in Schwarzschild-like coordinates making

$$\sinh \rho \rightarrow \frac{r}{l} ; \quad t \rightarrow \frac{t}{l} , \quad (5)$$

so that the metric reads

$$ds^2 = - \left(\alpha + \sqrt{\frac{r^2}{l^2} + 1} \right)^2 dt^2 + \frac{dr^2}{\frac{r^2}{l^2} + 1} + r^2 d\varphi^2 , \quad (6)$$

and the deviation with respect to the AdS metric at infinity is of the form $\Delta g_{tt} = -\frac{2\alpha}{l}r + \mathcal{O}(1)$. Note that the deviation grows instead of decaying as one approaches to the asymptotic region since it is linearly divergent, in sharp contrast with the standard behavior, given by $\Delta g_{tt} = \mathcal{O}(1)$ [31]. In spite of this divergent behavior, finite charges as surface integrals can be constructed through standard perturbative methods [32, 33]. However, in this case, quadratic deviations with respect to the background metric turn out to be relevant since they give additional nontrivial contributions to the surface integrals. The purpose of the next subsection is to compute the mass of the soliton within a fully nonlinear approach.

A. Soliton mass and Euclidean action from the Brown-York approach with suitable counterterms

Here it is shown that the soliton (3) possesses a negative fixed mass which coincides with the one of AdS spacetime, so that the integration constant α corresponds to a modulus parameter. The approach we follow is the quasilocal one of Brown and York [34], where the action is regularized with suitable counterterms along the lines of [35, 36]. For the BHT massive gravity theory, this task was performed by Hohm and Tonni [37] for a generic value of the parameter m^2 , and in the special case (2) by Giribet and Leston [38]. Thus, in our case, the suitable regularized action turns out to be given by

$$I_{\text{reg}} = I_{\text{BHT}} + I_{\text{GH}} + I_{\text{ct}} . \quad (7)$$

It is useful expressing the bulk action in second-order form by means of the auxiliary field $f^{\mu\nu}$, so that it reads

$$I_{\text{BHT}} = \frac{1}{16\pi G} \int_M d^3x \sqrt{-g} \left[R - 2\lambda - f^{\mu\nu} G_{\mu\nu} + \frac{1}{4} m^2 (f_{\mu\nu} f^{\mu\nu} - f^2) \right] , \quad (8)$$

and the boundary terms can be written as

$$I_{\text{GH}} = \frac{1}{16\pi G} \int_{\partial M} d^2x \sqrt{-\gamma} \left[-2K - \hat{f}^{ij} K_{ij} + \hat{f} K \right] , \quad (9)$$

$$I_{\text{ct}} = \frac{1}{16\pi G l} \int_{\partial M} d^2x \sqrt{-\gamma} \hat{f} , \quad (10)$$

where \hat{f}^{ij} is defined in terms of f^{ij} and the shift N^j of a radial ADM decomposition of the bulk metric according to

$$\hat{f}^{ij} = f^{ij} + 2f^{r(i} N^{j)} + f^{rr} N^i N^j . \quad (11)$$

It is also useful to express the boundary metric of ∂M in the standard ADM foliation with spacelike surfaces Σ , i.e.,

$$\gamma_{ij} dx^i dx^j = -N_\Sigma^2 dt^2 + \sigma(d\varphi + N_\Sigma^\varphi dt)(d\varphi + N_\Sigma^\varphi dt) , \quad (12)$$

so that the Brown-York stress tensor [34] is obtained by varying the regularized action with respect to the boundary metric γ_{ij}

$$T^{ij} = \lim_{r \rightarrow \infty} \frac{2}{\sqrt{-\gamma}} \frac{\delta I_{\text{reg}}}{\delta \gamma_{ij}} . \quad (13)$$

Therefore, the corresponding conserved charge associated to a Killing vector ξ is given by

$$Q(\xi) = \int_\Sigma d\varphi \sqrt{\sigma} u^i \xi^j T_{ij} , \quad (14)$$

where u^i is the timelike unit normal to Σ .

In the case of the soliton metric (3), the Brown-York stress-energy tensor reads

$$T_{ij}^{\text{sol}} = \begin{pmatrix} -\frac{1}{8\pi l G} & 0 \\ 0 & -\frac{l}{8\pi G} \end{pmatrix} , \quad (15)$$

and hence, its mass is finite, negative, and given by

$$M_{\text{sol}} = Q(\partial_t) = -\frac{1}{4G} . \quad (16)$$

Note that since the soliton mass does not depend on the parameter α , it exactly coincides with the one of AdS spacetime. Therefore, as this integration constant plays no role in the conserved charges, the soliton can be naturally regarded as a degenerate ground state labeled by a single modulus parameter.

The Euclidean continuation of the soliton can be easily obtained through $t \rightarrow i\tau_E$, where τ_E is the “Euclidean time” with an arbitrary period β , and since it possesses finite Euclidean action it describes an instanton. This can be explicitly seen as follows. Evaluating the Euclidean continuation of (6) one obtains that the relevant terms are given by

$$I_{\text{BHT}} = -\frac{\beta}{2G} \left(\frac{r^2}{l^2} \right) , \quad I_{\text{GH}} = \frac{\beta}{G} \left(\frac{r^2}{l^2} + \frac{1}{2} \right) , \quad I_{\text{ct}} = -\frac{\beta}{2G} \left(\frac{r^2}{l^2} + \frac{1}{2} \right) , \quad (17)$$

so that the divergences exactly cancel out and the total Euclidean action (7) for the soliton becomes finite and given by

$$I_{\text{sol}} = -\beta M_{\text{sol}} = \frac{1}{4G} \beta . \quad (18)$$

As expected, the Euclidean action does not depend on the modulus parameter α , and therefore it coincides with that of AdS spacetime. As a cross check, the soliton mass can alternatively be computed according to

$$M_{\text{sol}} = -\partial_\beta I_{\text{reg}} = -\frac{1}{4G} , \quad (19)$$

which precisely agrees with the result found above in (16) by means of the Brown-York stress-energy tensor.

III. ROTATING HAIRY BLACK HOLE: REGULARIZED EUCLIDEAN ACTION AND THERMODYNAMICS

The regularized Euclidean action (7) of the rotating hairy black hole turns out to be finite and it reduces to the free energy in the semiclassical approximation. As naturally expected, it depends only on the Hawking temperature and the angular velocity of the horizon, β and Ω_+ , respectively, and it is then independent of the gravitational hair parameter. This can be seen as follows: The field equations at the special point (2) admit a solution whose metric is given by [13]

$$ds^2 = -G(r) dt^2 + \frac{dr^2}{F(r)} + 2N^\phi(r) dt d\phi + (r^2 + r_0^2) d\phi^2 , \quad (20)$$

where

$$F(r) := \frac{r^2}{l^2} - \frac{b}{l} \left(1 + \Xi^{-\frac{1}{2}}\right) r + \frac{b^2}{4} \left(1 + \Xi^{-\frac{1}{2}}\right)^2 - 4G\mathcal{M}\Xi^{\frac{1}{2}} , \quad (21)$$

$$G(r) := \frac{r^2}{l^2} - 2\frac{b}{l}\Xi^{-\frac{1}{2}}r + \frac{b^2}{4} \left(1 + 3\Xi^{-1}\right) - 2G\mathcal{M} \left(1 + \Xi^{\frac{1}{2}}\right) , \quad (22)$$

$$N^\phi(r) := -a \left(\frac{b}{l}\Xi^{-\frac{1}{2}}r + 2G\mathcal{M} + \frac{b^2}{2}\Xi^{-1} \right) , \quad (23)$$

and the constants r_o^2 , and Ξ are defined as

$$r_o^2 := \frac{b^2 l^2}{4} (1 - \Xi^{-1}) + 2G\mathcal{M}l^2 \left(1 - \Xi^{\frac{1}{2}}\right) ,$$

$$\Xi := 1 - \frac{a^2}{l^2} ,$$

The solution depends on three integration constants, where \mathcal{M} corresponds to the mass, the angular momentum is given by $\mathcal{J} = \mathcal{M}a$ (with $a^2 \leq l^2$), and b is the gravitational hair parameter¹. Note that the BTZ black hole [39, 40] is recovered for $b = 0$. The singularity at $r = r_s := -\frac{1}{2}bl \left(1 - \Xi^{-\frac{1}{2}}\right)$ is cloaked by the event horizon located at $r = r_+$, with

$$r_+ = \frac{bl}{2} \left(1 + \Xi^{-\frac{1}{2}}\right) + 2l\sqrt{G\mathcal{M}\Xi^{\frac{1}{2}}} . \quad (24)$$

Cosmic censorship then requires that $r_+ \geq r_s$, which in the case of negative b , implies the following bound for the mass

$$\mathcal{M} \geq \frac{b^2 \Xi^{-\frac{1}{2}}}{4G} , \quad (25)$$

while for positive b the bound is such that mass turns out to be nonnegative.

The angular velocity of the horizon is given by

$$\Omega_+ = \frac{1}{a} \left(\Xi^{\frac{1}{2}} - 1 \right) ,$$

and the Hawking temperature expressed in terms of the Euclidean time period, $T = \beta^{-1}$, reads

$$\beta^2 = \frac{\pi^2 l^2}{2G\mathcal{M}} \left(1 + \Xi^{\frac{1}{2}}\right) \Xi^{-1} , \quad (26)$$

where the Euclidean continuation of the rotating black hole is performed through $t \rightarrow it_E$, and $a \rightarrow ia$.

¹ For simplicity, here the gravitational hair parameter b has been redefined making $b \rightarrow -2bl^{-1}$ in [13].

Evaluating the Euclidean rotating hairy black hole on each term of the regularized action (7), one obtains

$$\begin{aligned} I_{\text{BHT}} &= -\frac{\beta}{2G} \left(\frac{r^2}{l^2} - \frac{b}{l} r \left(1 + \Xi^{-\frac{1}{2}} \right) + \frac{b^2}{4} \left(1 + \Xi^{\frac{1}{2}} \right)^2 \Xi^{-1} - 4G\mathcal{M}\Xi^{\frac{1}{2}} \right) , \\ I_{\text{GH}} &= \frac{\beta}{G} \left(\frac{r^2}{l^2} - \frac{b}{l} r \left(1 + \Xi^{-\frac{1}{2}} \right) + \frac{b^2}{4} \left(1 + \Xi^{\frac{1}{2}} \right)^2 \Xi^{-1} - 2G\mathcal{M}\Xi^{\frac{1}{2}} \right) , \\ I_{\text{ct}} &= -\frac{\beta}{2G} \left(\frac{r^2}{l^2} - \frac{b}{l} r \left(1 + \Xi^{-\frac{1}{2}} \right) + \frac{b^2}{4} \left(1 + \Xi^{\frac{1}{2}} \right)^2 \Xi^{-1} - 2G\mathcal{M}\Xi^{\frac{1}{2}} \right) . \end{aligned}$$

Hence, the divergent terms cancel out so that the total Euclidean action (7) of the rotating hairy black hole becomes finite and given by

$$I_{\text{hbh}} = \beta \mathcal{M} \Xi^{\frac{1}{2}} . \quad (27)$$

As explained in [13, 41], since the gravitational hair parameter b is not related to a global charge, no chemical potential can be associated with. This can be independently confirmed by virtue of Eq. (27) since once expressed in terms of the non extensive variables β and Ω_+ , it reads

$$I_{\text{hbh}} = \frac{\pi^2 l^2}{G} \frac{1}{\beta} \left(\frac{1}{1 + \Omega_+^2 l^2} \right) , \quad (28)$$

which is manifestly independent of b .

The Euclidean action reduces to the free energy in the semiclassical approximation, i.e., $I_{\text{hbh}} = -\beta F$, with $F = \mathcal{M} - TS - \Omega_+ \mathcal{J}$, so that the first law is recovered requiring it to have an extremum. Indeed, the mass and the angular momentum are given by

$$\mathcal{M} = (-\partial_\beta + \beta^{-1} \Omega_+ \partial_\Omega) I_{\text{hbh}} , \quad (29)$$

$$\mathcal{J} = \beta^{-1} \partial_\Omega I_{\text{hbh}} , \quad (30)$$

which coincide with the expressions found through the evaluation of the Brown-York stress-energy tensor in [38]. This is also in full agreement with previous results [41]. In the static case, the mass has also been recovered by different methods in Refs. [42, 43]. Analogously, the black hole entropy can be obtained from

$$S = (1 - \beta \partial_\beta) I_{\text{hbh}} ,$$

which gives

$$S = \pi l \sqrt{\frac{2\mathcal{M}}{G} \left(1 + \Xi^{\frac{1}{2}} \right)} , \quad (31)$$

and agrees with the result found in [13, 41] by means of Wald's formula [44]. Further aspects concerning the rotating hairy black hole thermodynamics have been studied in Refs. [45, 46].

IV. MODULAR INVARIANCE, SOLITONS AND MICROSCOPIC ENTROPY OF THE ROTATING HAIRY BLACK HOLE

It is useful to express the Euclidean action of the soliton (18) and the hairy black hole (28) in terms of the modular parameter of the torus geometry at the boundary, given by

$$\tau := \frac{i\hat{\beta}}{2\pi l} , \quad (32)$$

with $\hat{\beta} := \beta(1 - i\Omega_+ l)$. The corresponding Euclidean actions then read

$$I_{\text{sol}} = i\pi l M_{\text{sol}} (\tau - \bar{\tau}) , \quad (33)$$

and

$$I_{\text{hbbh}} = -i\pi l M_{\text{sol}} \left(\frac{1}{\tau} - \frac{1}{\bar{\tau}} \right) , \quad (34)$$

respectively, where $M_{\text{sol}} = -\frac{1}{4G}$ stands for the soliton mass previously obtained in Eqs. (16) and (19). Note that (33) and (34) are related by a modular transformation given by

$$\tau \rightarrow -\frac{1}{\tau} , \quad (35)$$

as it is the case of Euclidean AdS and the BTZ black hole in General Relativity [48]. It is worth pointing out that the Euclidean action of the hairy black hole is completely determined by the soliton mass M_{sol} and the modular parameter τ , in full agreement with what occurs for black holes with scalar hair and scalar solitons in General Relativity [49, 52].

Remarkably, although the Euclidean actions (33) and (34) do not depend neither on the modulus parameter α of the soliton, nor on the gravitational hair parameter b of the black hole, since both solutions are dual under modular invariance, the gravitational hair turns out to be determined by the quotient of α and $|\tau|$. This can be seen as follows: The holographic realization of modular invariance that connects both solutions, amounts to a coordinate transformation given by²

$$\begin{aligned} \phi &= \frac{1}{2} \left[(\tau + \bar{\tau}) \varphi + i \left(\frac{1}{\tau} - \frac{1}{\bar{\tau}} \right) \tau_E \right] , \\ t_E &= -\frac{l}{2} \left[i (\tau - \bar{\tau}) \varphi + \left(\frac{1}{\tau} + \frac{1}{\bar{\tau}} \right) \tau_E \right] , \end{aligned} \quad (36)$$

² Note that according to (36) the modular transformation (35) swaps the role of Euclidean time and the angular coordinate.

with

$$r = \frac{l}{|\tau|} \cosh \rho - \frac{bl}{4} \left(\frac{\tau - \bar{\tau}}{|\tau|} \right)^2, \quad (37)$$

which means that an Euclidean rotating hairy black hole with coordinates (t_E, r, ϕ) characterized by a modular and hair parameters τ and b , respectively, is diffeomorphic to the Euclidean continuation of the soliton in Eq. (3), with coordinates (τ_E, ρ, φ) and modulus parameter $\alpha = b|\tau|$. In turn, this naturally suggests that a soliton with modulus parameter α corresponds to the ground state of a hairy black hole whose gravitational hair parameter is given by

$$b = \frac{\alpha}{|\tau|}. \quad (38)$$

It is also worth highlighting that the relationship expressed by Eq. (38) exactly maps the mass bound required by cosmic censorship for the hairy black hole (25) with the condition (4) that guarantees the smoothness of the soliton.

A. Cardy formula and spontaneous fixing of ground state modulus parameter

Formula (38) further suggests that a black hole with hair parameter b is in a broken phase of the theory in which the modulus parameter of the ground state is spontaneously fixed. In fact, in this case, it can be seen that semiclassical entropy agrees with the microscopic counting of states according to Cardy formula [50], as it occurs for General Relativity [51]. Indeed, as emphasized in [52], it is useful to express Cardy's formula in terms of the shifted Virasoro operators, $\tilde{L}_0^\pm := L_0^\pm - \frac{c^\pm}{24}$, so that it reads

$$S = 4\pi\sqrt{-\tilde{\Delta}_0^+ \tilde{\Delta}^+} + 4\pi\sqrt{-\tilde{\Delta}_0^- \tilde{\Delta}^-}, \quad (39)$$

where $(\tilde{\Delta}_0^\pm)$ $\tilde{\Delta}^\pm$ correspond to the (lowest) eigenvalues of \tilde{L}_0^\pm . Thus, noteworthy, the asymptotic growth of the number of states can be obtained only in terms of the spectrum of \tilde{L}_0^\pm without making any explicit reference to the central charges c^\pm .

The rotating hairy black hole entropy can then be computed from (39) assuming that the eigenvalues of \tilde{L}_0^\pm are given by the global charges of the black hole, i.e.,

$$\tilde{\Delta}^\pm = \frac{1}{2}(\mathcal{M}l \pm \mathcal{J}), \quad (40)$$

where the the ground state corresponds to the soliton (3), so that the lowest eigenvalues of

\tilde{L}_0^\pm are given by

$$\tilde{\Delta}_0^+ = \tilde{\Delta}_0^- = \frac{l}{2} M_{\text{sol}} = -\frac{l}{8G} .$$

Since in the case under consideration, the lowest eigenvalues $\tilde{\Delta}_0^\pm$ can be expressed in terms of the central charges³, i.e., $\tilde{\Delta}_0^\pm = -\frac{c^\pm}{24}$, one verifies that formula (39) reduces to its standard form

$$S = 2\pi\sqrt{\frac{c^+}{6}\tilde{\Delta}^+} + 2\pi\sqrt{\frac{c^-}{6}\tilde{\Delta}^-} , \quad (41)$$

which, as explained in [41] exactly reproduces the semiclassical entropy of the rotating hairy black hole in Eq. (31).

Note that in formula (39) it has been implicitly assumed that the ground state is non degenerate; otherwise, the asymptotic growth of the number of states would be given by

$$\rho(\tilde{\Delta}^\pm) = \rho(\tilde{\Delta}_0^\pm) \exp\left(4\pi\sqrt{-\tilde{\Delta}_0^\pm\tilde{\Delta}^\pm}\right) , \quad (42)$$

where $\rho(\tilde{\Delta}_0^\pm)$ correspond to the ground state degeneracy (see, e.g., [52–54]). Therefore, since formula (39) exactly matches the semiclassical entropy of the hairy black hole, one obtains that $\rho(\tilde{\Delta}_0^\pm) = 1$, which means that the ground state degeneracy is removed. This can be interpreted as the fact that the modulus parameter of the ground state is spontaneously fixed, so that the hairy black hole is actually in a broken phase of the theory. The purpose of the next section is to show that this is the case.

V. HAIRY BLACK HOLE-SOLITON PHASE TRANSITION

According to Eqs. (33) and (34), at high temperatures the partition function is dominated by the hairy black hole, while for low temperatures it turns out to be dominated by thermal radiation on the soliton. In order to see whether there is a phase transition involving these objects one has to identify different possible configurations with the same fixed modular parameter τ , i.e. one has to look for more than one configuration at fixed temperature β and chemical potential Ω_+ . It is simple to verify that this only occurs in the static case, where it can be seen that at fixed temperature there exists a phase transition between the

³ As shown in [13], at the special point (2) the central charges are given by twice the value of Brown and Henneaux, i.e., $c^+ = c^- = \frac{3l}{G}$.

hairy black hole and the soliton. Indeed, according to (28), in the case of the static hairy black hole the free energy is given by

$$F_{\text{hbh}} = -\frac{\pi^2 l^2}{G} T^2, \quad (43)$$

while for the soliton, from Eq. (18), the free energy turns out to be

$$F_{\text{sol}} = -\frac{1}{4G}. \quad (44)$$

Therefore, at the critical temperature

$$T = T_c := \frac{1}{2\pi l}, \quad (45)$$

which corresponds to the self-dual point of the modular transformation (35), the soliton and the hairy black hole possess the same free energy. This means that below the critical point ($T < T_c$), the soliton has less free energy than the hairy black hole, while for $T > T_c$ the hairy black hole is the configuration that dominates the partition function (See Fig. 1). Note that there is another possible decay channel at fixed β , corresponding to the extremal black hole. However, since its free energy vanishes, the extremal hairy black hole then always becomes unstable against thermal decay.

Since the first derivative of the free energy has a discontinuity at the critical temperature (45), given by

$$\left. \frac{\partial F(T)}{\partial T} \right|_+ - \left. \frac{\partial F(T)}{\partial T} \right|_- = -\frac{\pi l}{G}, \quad (46)$$

the phase transition between the hairy black hole and the soliton is of first order. As shown below, the presence of gravitational hair induces an additional effective length scale which might allow a suitable treatment of this phase transition in the semiclassical regime.

A. Gravitational hair, thermal fluctuations and phase transition in the semiclassical regime

In the static case the hairy black hole metric (20) acquires a very simple form, given by⁴

$$ds^2 = -\frac{1}{l^2} (r - r_+) (r - r_-) dt^2 + \frac{l^2}{(r - r_+) (r - r_-)} dr^2 + r^2 d\phi^2, \quad (47)$$

⁴ This metric was first found in the context of conformal gravity in three dimensions [55], and for BHT massive gravity it was independently discussed in [13] and [4].

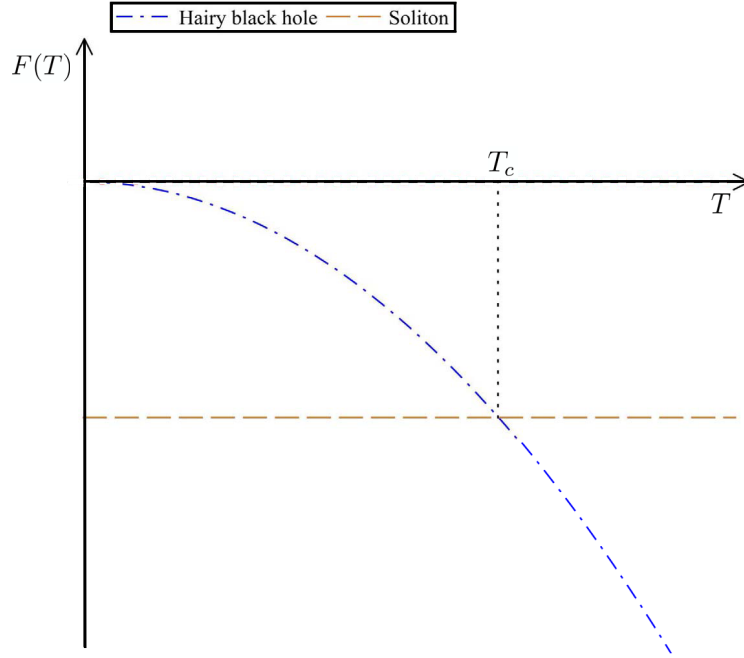


FIG. 1: Free energy as a function of the temperature for the soliton and the hairy black hole. Here $T_c = (2\pi l)^{-1}$.

where the integration constants, $r_- < r_+$, can be expressed in terms of the mass and the hair parameter according to

$$\mathcal{M} = \frac{1}{16Gl^2} (r_+ - r_-)^2, \quad (48)$$

$$b = \frac{1}{2l} (r_- + r_+). \quad (49)$$

The solutions splits in two branches according to the sign of b , such that for $b < 0$ there is a single event horizon located at $r = r_+$, provided the mass parameter fulfills the bound (25) with $a = 0$. In the case of $b > 0$ the constants r_- and r_+ correspond to the Cauchy and the event horizons respectively. Hence, in this case the hair parameter introduces an effective length scale that corresponds to the horizon radius of the extremal black hole, given by $r_+ = r_- = r_e$, with

$$r_e := bl, \quad (50)$$

that fixes the minimum size of the hairy black hole ($r_+ \geq r_e$).

In order to explore the nature of the phase transition it is useful looking at the behaviour of the specific heat $C = \frac{d\mathcal{M}}{dT}$, which for the static hairy black hole (47) is given by

$$C = \frac{2\pi^2 l^2}{G} T = \frac{\pi}{G} (r_+ - r_e) . \quad (51)$$

Since the specific heat (51) is positive for a non extremal black hole, it can reach local thermal equilibrium with a heat bath provided the thermal fluctuations are small. Note that for the near extremal black hole, $r_+ \gtrsim r_e$, the heat capacity becomes arbitrarily small, and therefore a fixed amount of energy that is either absorbed or radiated by the black hole, necessarily implies a large fluctuation of the temperature. In this sense, the near extremal black hole becomes “volatile”, which signals the existence of the phase transition being triggered by thermal fluctuations. Indeed, the energy and temperature fluctuations are given by⁵

$$\frac{(\Delta E)^2}{E^2} = \frac{4}{\pi} \frac{\ell_p}{(r_+ - r_e)} , \quad (52)$$

$$\frac{(\Delta T)^2}{T^2} = \frac{\ell_p}{\pi (r_+ - r_e)} , \quad (53)$$

respectively, where $\ell_p := G$ is the Planck length. Since for a near extremal black hole the difference $r_+ - r_e$ is very small, according to Eqs. (52) and (53), in this case the energy and temperature fluctuations become very large, and hence hairy black holes turn out to be thermodynamically unstable at low temperatures.

Remarkably, due to the existence of gravitational hair, which fixes the size of the extremal hairy black hole as in Eq. (50), the transition is able to take place in the semiclassical regime of the theory which ensures the reliability of the previous analysis. This is because in the semiclassical approximation the event horizon, extremal, and AdS radii have to be much larger than the Planck length, i.e., $l, r_+, r_e \gg \ell_p$. This means that the growth of the thermal fluctuations for a low temperature hairy black hole can be seen occur in the semiclassical approximation, provided the scale introduced by the gravitational hair, which fixes the radius of the extremal black hole fulfills $r_e \gg \ell_p$. Note that this is not the case for the BTZ black hole, for which $r_e = 0$, because in the semiclassical regime $r_+ \gg \ell_p$ the fluctuations become very small, so that the possible phase transition would occur in a regime where quantum gravity effects become relevant (see e.g. [56–59]).

⁵ In the canonical ensemble the fluctuations are related to the specific heat as $(\Delta E)^2 = CT^2$, and $(\Delta T)^2 = C^{-1}T^2$.

As an ending remark, it is worth pointing out that a usual feature of first order phase transitions, as it occurs for water, is the possibility of bubble nucleation due to the existence of metastable states. Indeed, apart from the critical temperature T_c defined in Eq. (45), it is useful to introduce a temperature T_1 , given by

$$T_1 = \frac{4}{\pi} \left(\frac{\ell_p}{l} \right) T_c, \quad (54)$$

for which the energy fluctuations (52) are of order one. Thus, for $T > T_c$ the black hole clearly dominates the partition function, while for $T_1 < T < T_c$ the black hole turns out to be metastable against vacuum (soliton) nucleation, since it possesses less free energy than the soliton but the thermal fluctuations are too small so as to trigger a sudden transition. For $T < T_1$ the vacuum in a thermal bath of radiation is the preferred configuration. Note that in the transition from the ground state to the black hole, the size that characterizes an extremal black hole, determined by the hair parameter, would be spontaneously chosen as the temperature increases.

VI. SUMMARY AND DISCUSSION

The purely gravitational soliton described by the metric (3) was shown to possess a fixed negative mass M_{sol} given by (16) which does not depend on the integration constant α . Thus, the soliton can be naturally regarded as a degenerate ground state labeled by a single modulus parameter, whose mass precisely coincides with the one of AdS spacetime. According to Eqs. (18) and (27), the Euclidean action (7) was shown to be finite and independent of modulus and hair parameters for the soliton as well as for the hairy black hole (20). It is then amusing to verify that the hair parameter b , an integration constant that cannot be gauged away since it has an apparent effect in the causal structure of the hairy black hole, has a missing role in the global charges. The corresponding masses (19) and (29), with the angular momentum (30) and the entropy (31) in the case of the black hole were successfully recovered from a different method. It was also shown that the Euclidean actions of the soliton and the rotating hairy black hole can be written as in Eqs. (33) and (34), respectively, so that they are completely determined by the soliton mass M_{sol} and the modular parameter τ in Eq. (32), as it occurs for a different class of black holes with scalar hair and scalar solitons in General Relativity [49, 52]. Both Euclidean actions turned

out to be related by a modular transformation given by (35). This is in full agreement with the case of Euclidean AdS and the BTZ black hole in General Relativity [48], which is a consequence of the fact that both Euclidean solutions are diffeomorphic [60]. In this sense, here the soliton plays the role of AdS in General Relativity. Indeed, according to Eqs. (36) and (37), the Euclidean rotating black hairy hole becomes diffeomorphic to the Euclidean soliton, provided the quotient of the modulus and hair parameter is fixed by the norm of τ as in Eq. (38), which remarkably maps the mass bound required by cosmic censorship for the hairy black hole (25) with the condition (4) that guarantees the regularity of the soliton at the origin. The relationship (38) further suggests that the hairy black hole is in a broken phase of the theory, in which the modulus parameter of the ground state is spontaneously fixed. This is supported by the fact that Cardy formula (39) agrees with the semiclassical entropy (31) when the degeneracy of the ground state is removed (see Eq. (42)). It is then reassuring to verify that there is a critical temperature $T_c = (2\pi l)^{-1}$ characterizing a phase transition between the static hairy black hole and the soliton. This phase transition is of first order and it turns out to be qualitatively different than the one of Hawking and Page [61] between the Schwarzschild-AdS black hole and AdS spacetime (for a recent discussion see [47]). Moreover, the existence of gravitational hair parameter induces an additional effective length scale that determines the minimum size of the black hole (see Eq. (50)) which allows a suitable treatment of this phase transition in the semiclassical regime.

One may speculate that the spontaneous choice of the modulus parameter α , which determines the gravitational hair, could be related with some sort of symmetry breaking mechanism. Indeed, at the special point (2) the linearized graviton becomes partially massless, possessing an additional gauge symmetry which makes it to possess only one degree of freedom. This symmetry is certainly broken by the self interactions around a generic configuration, but it may survive around certain classes of solutions that includes the soliton. Preliminary results indicate that there is an enhancement of gauge symmetries around certain configurations at the full nonlinear level [62], as it has been observed for different classes of degenerate dynamical systems (see e.g. [63]). Simple examples of this phenomenon exist in classical mechanics [64], for which the rank of the symplectic form may decrease on certain regions within the space of configurations, so that around certain special classes of solutions, additional gauge symmetries arise and then the system losses some degrees of freedom. Unusual mechanisms like this one are clearly out of the hypotheses of the Coleman-

Mermin-Wagner theorem [65] (see e.g. [66]), which also appears to be circumvented in the context of holographic superconductors in $1 + 1$ dimensions [67–69].

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- [1] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982).
 - [2] S. Deser, R. Jackiw and S. Templeton, Annals Phys. **140**, 372 (1982) [Erratum-ibid. **185**, 406 (1988)] [Annals Phys. **185**, 406 (1988)] [Annals Phys. **281**, 409 (2000)].
 - [3] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. Lett. **102**, 201301 (2009) [arXiv:0901.1766 [hep-th]].
 - [4] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. D **79**, 124042 (2009) [arXiv:0905.1259 [hep-th]].
 - [5] M. Blagojevic and B. Cvetkovic, JHEP **1101**, 082 (2011) [arXiv:1010.2596 [gr-qc]].
 - [6] M. Blagojevic and B. Cvetkovic, JHEP **1103**, 139 (2011) [arXiv:1103.2388 [gr-qc]].
 - [7] M. Sadegh and A. Shirzad, Phys. Rev. D **83**, 084040 (2011) [arXiv:1010.2887 [hep-th]].
 - [8] B. Tekin, “Partially massless spin-2 fields in string generated models,” arXiv:hep-th/0306178.
 - [9] S. Deser and R. I. Nepomechie, Annals Phys. **154**, 396 (1984).
 - [10] S. Deser and A. Waldron, Phys. Rev. Lett. **87**, 031601 (2001) [arXiv:hep-th/0102166].
 - [11] S. Deser and A. Waldron, Nucl. Phys. B **607**, 577 (2001) [arXiv:hep-th/0103198].
 - [12] S. Deser, Phys. Rev. Lett. **103**, 101302 (2009) [arXiv:0904.4473 [hep-th]].
 - [13] J. Oliva, D. Tempo and R. Troncoso, JHEP **0907**, 011 (2009) [arXiv:0905.1545 [hep-th]].
 - [14] G. Clement, Class. Quant. Grav. **26**, 105015 (2009) [arXiv:0902.4634 [hep-th]].
 - [15] G. Clement, Class. Quant. Grav. **26**, 165002 (2009) [arXiv:0905.0553 [hep-th]].
 - [16] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, Phys. Rev. D **80**, 104029 (2009) [arXiv:0909.1347 [hep-th]].

- [17] E. Ayon-Beato, G. Giribet and M. Hassaine, JHEP **0905**, 029 (2009).
- [18] M. Chakhad, “Kundt spacetimes of massive gravity in three dimensions,” arXiv:0907.1973 [hep-th].
- [19] H. Ahmedov and A. N. Aliev, Phys. Lett. B **694**, 143 (2010) [arXiv:1008.0303 [hep-th]].
- [20] H. L. C. Louzada, U. C. dS and G. M. Sotkov, Phys. Lett. B **686**, 268 (2010) [arXiv:1001.3622 [hep-th]].
- [21] U. d. Camara and G. M. Sotkov, Phys. Lett. B **694** (2010) 94 [arXiv:1008.2553 [hep-th]].
- [22] H. Ahmedov and A. N. Aliev, Phys. Rev. D **83**, 084032 (2011) [arXiv:1103.1086 [hep-th]].
- [23] H. Maeda, JHEP **1102** (2011) 039 [arXiv:1012.5048 [hep-th]].
- [24] I. Bakas and C. Sourdis, Class. Quant. Grav. **28**, 015012 (2011) [arXiv:1006.1871 [hep-th]].
- [25] A. Ghodsi and M. Moghadassi, Phys. Lett. B **695**, 359 (2011) [arXiv:1007.4323 [hep-th]].
- [26] I. Gullu, T. C. Sisman and B. Tekin, Phys. Rev. D **82** (2010) 024032 [arXiv:1005.3214 [hep-th]].
- [27] A. Sinha, JHEP **1006** (2010) 061 [arXiv:1003.0683 [hep-th]].
- [28] D. Grumiller, N. Johansson and T. Zojer, JHEP **1101**, 090 (2011) [arXiv:1010.4449 [hep-th]].
- [29] A. Sinha, Class. Quant. Grav. **28**, 085002 (2011) [arXiv:1008.4315 [hep-th]].
- [30] Y. Kwon, S. Nam, J. D. Park and S. H. Yi, “Quasi Normal Modes for New Type Black Holes in New Massive Gravity,” arXiv:1102.0138 [hep-th].
- [31] J. D. Brown and M. Henneaux, Commun. Math. Phys. **104**, 207 (1986).
- [32] L. F. Abbott and S. Deser, Nucl. Phys. B **195**, 76 (1982).
- [33] S. Deser and B. Tekin, Phys. Rev. D **67**, 084009 (2003) [arXiv:hep-th/0212292].
- [34] J. D. Brown and J. W. York, Phys. Rev. D **47**, 1407 (1993) [arXiv:gr-qc/9209012].
- [35] M. Henningson and K. Skenderis, JHEP **9807**, 023 (1998) [arXiv:hep-th/9806087].
- [36] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208**, 413 (1999) [arXiv:hep-th/9902121].
- [37] O. Hohm and E. Tonni, JHEP **1004**, 093 (2010) [arXiv:1001.3598 [hep-th]].
- [38] G. Giribet and M. Leston, JHEP **1009**, 070 (2010) [arXiv:1006.3349 [hep-th]].
- [39] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992) [arXiv:hep-th/9204099].
- [40] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D **48**, 1506 (1993) [arXiv:gr-qc/9302012].
- [41] G. Giribet, J. Oliva, D. Tempo and R. Troncoso, Phys. Rev. D **80**, 124046 (2009)

- [arXiv:0909.2564 [hep-th]].
- [42] S. Nam, J. D. Park and S. H. Yi, Phys. Rev. D **82**, 124049 (2010) [arXiv:1009.1962 [hep-th]].
 - [43] S. Nam, J. D. Park and S. H. Yi, JHEP **1007** (2010) 058 [arXiv:1005.1619 [hep-th]].
 - [44] R. M. Wald, Phys. Rev. D **48**, 3427 (1993) [arXiv:gr-qc/9307038].
 - [45] B. Mirza and Z. Sherkatghanad, Phys. Rev. D **83**, 104001 (2011) [arXiv:1104.0390 [gr-qc]].
 - [46] R. Li, S. Li and J. R. Ren, Class. Quant. Grav. **27**, 155011 (2010) [arXiv:1005.3615 [hep-th]].
 - [47] R. Banerjee, S. K. Modak and D. Roychowdhury, arXiv:1106.3877 [gr-qc].
 - [48] J. M. Maldacena and A. Strominger, JHEP **9812**, 005 (1998) [arXiv:hep-th/9804085].
 - [49] M. Henneaux, C. Martinez, R. Troncoso and J. Zanelli, Phys. Rev. D **65**, 104007 (2002) [arXiv:hep-th/0201170].
 - [50] J. L. Cardy, Nucl. Phys. B **270**, 186 (1986).
 - [51] A. Strominger, JHEP **9802**, 009 (1998) [arXiv:hep-th/9712251].
 - [52] F. Correa, C. Martinez and R. Troncoso, JHEP **1101** (2011) 034 [arXiv:1010.1259 [hep-th]].
 - [53] S. Carlip, Class. Quant. Grav. **16**, 3327 (1999) [arXiv:gr-qc/9906126].
 - [54] F. Loran, M. M. Sheikh-Jabbari and M. Vincon, JHEP **1101**, 110 (2011) [arXiv:1010.3561 [hep-th]].
 - [55] J. Oliva, D. Tempo and R. Troncoso, Int. J. Mod. Phys. A **24**, 1588 (2009) [arXiv:0905.1510 [hep-th]].
 - [56] N. Cruz and S. Lepe, Phys. Lett. B **593**, 235 (2004) [arXiv:hep-th/0404218].
 - [57] Y. Kurita and M. a. Sakagami, Prog. Theor. Phys. **113** (2005) 1193 [arXiv:hep-th/0403091].
 - [58] B. Reznik, Phys. Rev. D **51** (1995) 1728 [arXiv:gr-qc/9403027].
 - [59] R. G. Cai, Z. J. Lu and Y. Z. Zhang, Phys. Rev. D **55** (1997) 853 [arXiv:gr-qc/9702032].
 - [60] S. Carlip and C. Teitelboim, Phys. Rev. D **51**, 622 (1995) [arXiv:gr-qc/9405070].
 - [61] S. W. Hawking and D. N. Page, Commun. Math. Phys. **87**, 577 (1983).
 - [62] A. Pérez, D. Tempo and R. Troncoso, work in progress.
 - [63] M. Banados, L. J. Garay and M. Henneaux, Nucl. Phys. B **476**, 611 (1996)[arXiv:hep-th/9605159]; O. Chandia, R. Troncoso and J. Zanelli, AIP Conf. Proc. **484**, 231 (1999) [arXiv:hep-th/9903204]; O. Miskovic, R. Troncoso and J. Zanelli, Phys. Lett. B **615**, 277 (2005)[arXiv:hep-th/0504055]; O. Miskovic, R. Troncoso and J. Zanelli, Phys. Lett. B **637**, 317 (2006)[arXiv:hep-th/0603183].
 - [64] J. Saavedra, R. Troncoso and J. Zanelli, J. Math. Phys. **42**, 4383 (2001)[arXiv:hep-

- th/0011231].
- [65] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17** (1966) 1133; P. C. Hohenberg, Phys. Rev. **158**, 383 (1967); S. R. Coleman, Commun. Math. Phys. **31** (1973) 259.
 - [66] D. Anninos, S. A. Hartnoll and N. Iqbal, Phys. Rev. D **82** (2010) 066008 [arXiv:1005.1973 [hep-th]].
 - [67] J. Ren, JHEP **1011** (2010) 055 [arXiv:1008.3904 [hep-th]].
 - [68] N. Lashkari, “Holographic Symmetry-Breaking Phases in $\text{AdS}_3/\text{CFT}_2$,” arXiv:1011.3520 [hep-th].
 - [69] Y. Liu, Q. Pan and B. Wang, “Holographic superconductor developed in BTZ black hole background with backreactions,” arXiv:1106.4353 [hep-th].